Engineering Maths First Aid Kit

6.3

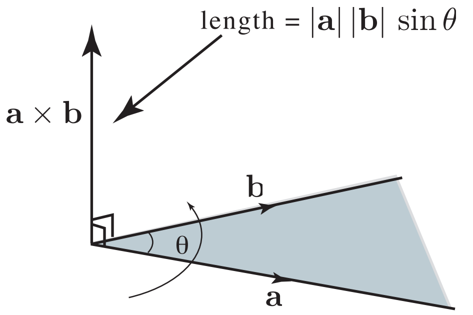
The vector product

Introduction

On this leaflet we describe how to find the **vector product** of two vectors.

1. Definition of the vector product

The result of finding the vector product of two vectors, $\mathbf{a}$ and $\mathbf{b}$, is a vector of modulus $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$ in the direction of $\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right-handed screw rule as shown below. The symbol used for the vector product is the times sign, $\times$. Do not use a dot, $\cdot$, because this is the symbol used for a scalar product.



$$\text{vector product:} \quad \mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{e}}$$

2. A formula for finding the vector product

A formula exists for finding the vector product of two vectors given in cartesian form:

If $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad$ and $\quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad$ then

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example

Evaluate the vector product $\mathbf{a}\times\mathbf{b}$ if $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$.

Solution

By inspection $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$, and so

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= ((-2)(-8) - (5)(4))\mathbf{i} - ((3)(-8) - (5)(7))\mathbf{j} + ((3)(4) - (-2)(7))\mathbf{k} \\ &= -4\mathbf{i} + 59\mathbf{j} + 26\mathbf{k}\end{aligned}$$

3. Using determinants to evaluate a vector product

Evaluation of a vector product using the previous formula is very cumbersome. There is a more convenient and easily remembered method for those of you who are familiar with determinants. The vector product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ can be found by evaluating the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

To find the $\mathbf{i}$ component of the vector product, imagine crossing out the row and column containing $\mathbf{i}$ and finding the determinant of what is left, that is

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

The resulting number is the $\mathbf{i}$ component of the vector product. The $\mathbf{j}$ component is found by crossing out the row and column containing $\mathbf{j}$ and evaluating

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1$$

<u>and then changing the sign of the result</u>. Finally the $\mathbf{k}$ component is found by crossing out the row and column containing $\mathbf{k}$ and evaluating

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example
Find the vector product of $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

Solution
The two given vectors are represented in the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$\mathbf{a} \times \mathbf{b} = (-8 - (-12))\mathbf{i} - (6 - 18)\mathbf{j} + (-18 - (-36))\mathbf{k} = 4\mathbf{i} + 12\mathbf{j} + 18\mathbf{k}$$

Exercises
1. If $\mathbf{a} = 8\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ show that $\mathbf{a} \times \mathbf{b} = -5\mathbf{i} - 18\mathbf{j} - 29\mathbf{k}$. Show also that $\mathbf{b} \times \mathbf{a}$ is not equal to $\mathbf{a} \times \mathbf{b}$, but rather that $\mathbf{b} \times \mathbf{a} = 5\mathbf{i} + 18\mathbf{j} + 29\mathbf{k}$.

